UNIFORM DOMAINS ON HOLOMORPHIC CURVES

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Abstract. We study a recent criterion for the injectivity of a holomorphic curve in $\Psi : \mathbb{D} \to \mathbb{C}^n$, the strong form of which is shown to imply that the image in $\mathbb{C}^n$ is a *uniform surface*. This concept is borrowed from geometric function theory in connection with quasidisks. We also obtain a two-point distortion theorem giving sharp estimates for the separation of images in terms of the hyperbolic distance in $\mathbb{D}$.

1. Introduction

In recent years, several injectivity criteria have been established for the conformal immersion of the unit disk $\mathbb{D} \subset \mathbb{C}$ into higher dimensional euclidean spaces. Important instances are represented by the Weierstrass-Enneper lift of a harmonic mapping and by a holomorphic curve in $\mathbb{C}^n$ parametrized by $\mathbb{D}$ [3], [4]. The criteria involve bounds on the Gaussian curvature of the image surface and on a Schwarzian derivative of the immersion stemming from conformal differential geometry. They constitute generalizations of classical conditions of Nehari in geometric function theory [10], with results that go beyond injectivity. In particular, the immersions are shown to admit a continuous extension to the closed disk with an analysis of the cases when such an extension can fail to be injective on the boundary. We cite [12] for a seminal paper analyzing the issues of continuous extension and extremal mappings for one of Nehari's condition.

In one way or another, all classical Schwarzian univalence criteria entail establishing the disconjugacy of solutions of an associated second order linear differential equation. The higher dimensional analogues do as well, but through an allied one-dimensional Schwarzian operator introduced by Ahlfors in his study of the distortion of cross-ratio and curves in euclidean spaces [1]. Of independent interest we mention sharp criteria for curves to be simple or even unknotted that can be derived in terms of this operator

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[7]. The appearance of the Gaussian curvature in the high dimensional criteria can be explained as the difference between the conformal Schwarzian of the immersion along a curve in the domain, and Ahlfors' derivative of the restriction of the immersion to that curve. We refer to [13] for a formulation of a derivative that encompasses both Ahlfors operator as well as the conformal Schwarzian, giving thus a unified approach to the various criteria here described.

An important phenomenon in classical theory is the connection between univalence criteria and quasiconformal mappings (see, *e.g.*, [2]). As a general rule, the strong form of a univalence condition in $\mathbb{D}$ (or any quasidisk) ensures that the mappings f considered admit a quasiconformal extension to the plane [12]. Equivalently, the image $f(\mathbb{D})$ is a quasidisk, a property that can be formulated entirely in terms of metric conditions of $f(\mathbb{D})$. Furthermore, this characterization can be translated to a property of the boundary $\partial f(\mathbb{D})$ by the well-known *four-point condition* of Ahlfors. In higher dimensions there are no known characteristic properties for a topological ball to be the image of a ball under a quasiconformal mapping of the entire space. Some results in this direction can be found in [14], [6], where the strong form of the criterion for the injectivity of Weierstrass-Enneper lifts $\tilde{f}$ is studied. Specifically, in [14] the author shows that the images $\tilde{f}(\mathbb{D})$ on the minimal surface, when bounded, are linearly-connected, John domains relative to the surface metric. In two dimensions, this would be equivalent to the image of the disk being a bounded quasidisk. In [6], the authors obtain a quasiconformal extension to 3-space of the Weierstrass-Ennerper lift under the same Schwarzian bound considered in [14]. Thus, the "quasidisk" $\tilde{f}(\mathbb{D})$ in [14] is a "hemisphere" of the quasisphere $\tilde{f}(\mathbb{C} \cup \{\infty\})$ that bounds the quasiball $\tilde{f}((\mathbb{R}^3)^+)$ in space. A connection between the 3-dimensional quasiconformal geometry of a quasi-ball and the 2-dimensional geometry of its boundary is unknown to us.

The purpose of the present paper is to extend the results in [14] for the strong form of the injectivity criterion for holomorphic curves in $\mathbb{C}^n$ parametrized by $\mathbb{D}$. For this purpose we introduce the notion of a *uniform surface* in complete analogy to the notion of a uniform domain that characterizes a planar quasidisk. The techniques developed here apply as well to the case treated in [14]. An improvement is that our results are valid regardless of the boundedness of the image. In addition, we establish sharp theorems for the separation of images in terms of the hyperbolic distance in $\mathbb{D}$.

The paper is organized as follows. In Section 2 we give a brief account of the standard and the conformal Schwarzian derivative, and a classical criterion of Nehari with its formulation for the case of a holomorphic curve. We present here the notion of a uniform surface, with the statements of our main results. Proofs are deferred to the last section. Sections 3 and 4 are devoted to Ahlfors operator and to important lemmas regarding the metric on the image surface in $\mathbb{C}^n$.

2. Preliminaries and Main Results

We recall the definition of Schwarzian derivative of an analytic and locally univalent function f given by

$$S_f = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2.$$

Nehari [10] proved that a function f analytic and locally univalent in $\mathbb{D} := \{z : |z| < 1\}$ is univalent if

$$|S_f(z)| \le 2p(|z|), \tag{1}$$

where $p : (-1,1) \to \mathbb{R}^+$ satisfies the conditions

(i) p is a continuous and even function;
(ii) $(1-t^2)^2 p(t)$ is decreasing in $(0,1)$;
(iii) the differential equation $u'' + pu = 0$ has no nontrivial solutions with more than one zero in $(-1,1)$.

A function p satisfying the conditions above will be called a Nehari p-function. An important special case constitutes the Nehari class N of functions satisfying the above criterion for $p(t) = (1-t^2)^{-2}$. This class has been the subject of several investigations in connection with quasiconformal mappings (see *e.g.*, [2], [12]). In [9] the authros derive other geometric and analytic properties of functions in the class N and showed, in particular, that

$$\left|\frac{f''(z)}{f'(z)}\right| \le 2\mu\frac{|z|}{1-|z|^2}, \qquad 0 < \mu \le 1 \tag{2}$$

for functions f satisfying(1) with $p(t) = \mu(1-t^2)^{-2}$ and $f''(0) = 0$. In Lemma 1 below we generalize (2) to a corresponding class of holomorphic curves.

A holomorphic curve is a smooth function Ψ defined from a domain $\Omega \subset \mathbb{C}$ into $\mathbb{C}^n$, $n \ge 1$, such that

$$\Psi'(z) := \lim_{h\to 0}\frac{\Psi(z+h)-\Psi(z)}{h}, \qquad h \in \mathbb{C}$$

exists for all $z \in \Omega$. It follows that if Ψ is a holomorphic curve and $\Psi = (\psi_1, \dots, \psi_n)$, then $\psi_k : \Omega \to \mathbb{C}$ is analytic for all $k = 1, \dots, n$. We are interested in the case $\Psi'(z) \ne 0$ for all $z \in \mathbb{D}$. Under this assumption Ψ is a locally injective holomorphic curve. We regard $\Sigma = \Psi(\mathbb{D})$ as a 2-dimensional surface in $\mathbb{C}^n \cong \mathbb{R}^{2n}$. The mapping Ψ represents a conformal parametrization of Σ with first fundamental form $\lambda(z)|dz|$ given by

$$\lambda^2 = \lambda_\Psi^2 = \|\Psi'\|^2 = |\psi_1'|^2 + \cdots + |\psi_n'|^2.$$

The Gaussian curvature at a point $\Psi(z)$ on Σ is given by

$$K(z) = -\frac{1}{\lambda^2(z)}\Delta \log \lambda(z). \tag{3}$$

Note that if f is a locally injective holomorphic function, then $\Psi \circ f$ is a holomorphic curve which satisfies $(\Psi \circ f)'(z) \neq 0$ for all z and

$$\lambda_{\Psi \circ f} = (\lambda_\Psi \circ f)|f'|. \tag{4}$$

The Schwarzian derivative of $\Psi : \mathbb{D} \to \mathbb{C}^n$ is defined in [4] by

$$S\Psi = 2(\partial_{zz}(\log \lambda) - (\partial_z \log \lambda)^2).$$

This reduces to the classical Schwarzian when $n = 1$. A straightforward calculation shows that

$$S(\Psi \circ f) = (S\Psi \circ f)(f')^2 + S_f,$$

where f is a locally injective holomorphic function. In particular, if T is a conformal automorphism of $\mathbb{D}$, we obtain

$$S(\Psi \circ T) = (S\Psi \circ T)(T')^2. \tag{5}$$

In [4] the authors prove the following criterion of univalence:

Theorem 1. *Let p be a Nehari function and $\Psi : \mathbb{D} \to \mathbb{C}^n$ a holomorphic curve such that $\Psi'(z) \neq 0$ for all $z \in \mathbb{D}$. If*

$$|S\Psi(z)| + \frac{3}{4}\lambda^2(z)|K(z)| \leq 2p(|z|), \qquad z \in \mathbb{D}, \tag{6}$$

then Ψ is injective and has a spherically continuous extension to the closure of $\mathbb{D}$. If we have strict inequality in (6) in a ring of the form $\{z : r_0 \leq |z| < 1\}$, then the extension is injective in $\overline{\mathbb{D}}$.

We will denote by Nh^μ the family of holomorphic curves satisfying

$$|S\Psi(z)| + \frac{3}{4}\lambda^2(z)|K(z)| \leq \frac{2\mu}{(1-|z|^2)^2}, \quad 0 < \mu \leq 1.$$

We will write Nh instead of Nh^1. It is easy to see that if $T \in \mathrm{Aut}(\mathbb{D})$ and $\Psi \in Nh^\mu$, then $\Psi \circ T \in Nh^\mu$. Also, Nh_0^μ will denote the family of holomorphic curves $\Psi \in Nh^\mu$ with $\partial_z \lambda(0) = 0$.

In this paper we will consider the following definition of uniform domain in analogy to the definition found in [11] for planar domains:

Definition 1. *A surface $S \subset \mathbb{R}^n$ is said to be an uniform surface if there exist constants a and b such that each pair of points x_1, $x_2 \in S$ can be joined by a rectifiable arc $\gamma \subset S$ for which*

(i) $l(\gamma) \leq a\|x_1 - x_2\|$;
(ii) $\min_{j=1,2} l(\gamma(x_j, x)) \leq b d(x, \partial S)$ *for all $x \in \gamma$.*

Here $l(\gamma)$ denotes the Euclidean length of γ, $\gamma(x_j, x)$ the part of γ between x_j and x, and $d(x, \partial S)$ stands for the extended real number

$$d(x, \partial S) := \sup\{ r \geq 0 : B_S(x, r) \subset S \},$$

where $B_S(x, r)$ is the ball in S centered at x with radius r.

The main purpose of this paper is to establish sufficient conditions for image of the unit disk under a holomorphic curve Ψ to be a uniform surface. We will also obtain a two-point distortion theorem associated with a holomorphic curve.

Further background is discussed in Section 3. Below, we summarize our main results.

Theorem 2. *Suppose $\Psi \in Nh^{\mu}$ and $0 < \mu < 1$. Then $\Psi(\mathbb{D})$ is an uniform surface.*

Theorem 3. *Let $\Psi \in Nh^{\mu}$ and $0 < \mu < 1$. Then*

$$\|\Psi(z_1) - \Psi(z_2)\| \geq \sqrt{(1 - |z_1|^2)\lambda(z_1)(1 - |z_2|^2)\lambda(z_2)}\, d_h(z_1, z_2) \tag{7}$$

for all $z_1, z_2 \in \mathbb{D}$. Moreover, if $\Psi \in Nh_0^{\mu}$ and $\lambda(0) = 1$, then

$$\|\Psi(z_1) - \Psi(z_2)\| \leq \frac{4\pi}{\sqrt{1-\mu}} |z_1 - z_2|^{\sqrt{1-\mu}} \tag{8}$$

for all $z_1, z_2 \in \mathbb{D}$.

A distortion theorem of type (8) for the family Nh was proven in [4]. The inequality (8) is an extension to holomorphic curves of a result for holomorphic functions obtained in [8]. The inequality (7) generalizes the result obtained in [5] for an analogous class of harmonic mappings, and our proof is based on the one given in [5] for the case $p(t) = \mu(1 - t^2)^{-2}$.

3. The Schwarzian derivative of Ahlfors

Ahlfors [1] defined the Schwarzian derivative of a regular curve $\varphi : (a, b) \to \mathbb{R}^n$ of class C^3, also called Ahlfors' Schwarzian, by

$$S_1\varphi = \frac{\langle \varphi', \varphi''' \rangle}{\|\varphi'\|^2} - 3\frac{\langle \varphi', \varphi'' \rangle^2}{\|\varphi'\|^4} + \frac{3}{2}\frac{\|\varphi''\|^2}{\|\varphi'\|^2},$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product in $\mathbb{R}^n$. Ahlfors' Schwarzian is invariant under post-composition with a Möbius transformation of $\mathbb{R}^n$ and satisfies a chain rule. More precisely, if $x : (c, d) \to (a, b)$ is of class C^3 with $x'(t) \neq 0$ for all $t \in (c, d)$, then

$$S_1(\varphi \circ x)(t) = S_1\varphi(x(t))x'(t)^2 + S_1x(t),$$

where

$$Sx := S_1x = \left(\frac{x''}{x'}\right)' - \frac{1}{2}\left(\frac{x''}{x'}\right)^2.$$

Chuaqui and Gevirtz [7] obtained an important expression for S_1 in terms of geometric quantities of the trace, namely

$$S_1\varphi = \left(\frac{v'}{v}\right)' - \frac{1}{2}\left(\frac{v'}{v}\right)^2 + \frac{1}{2}v^2k^2 = Ss + \frac{1}{2}v^2k^2, \tag{9}$$

where v is the velocity of φ, k its curvature, and s its arc length. In [7] the authors proved the following injectivity criterion for curves in $\mathbb{R}^n$.

Theorem 4. *Let P be a continuous function on $(-1,1)$ such that no non-trivial solution u of the differential equation $u'' + Pu = 0$ has more that one zero. Let $\varphi : (-1,1) \to \mathbb{R}^n \cup \{\infty\}$ be a regular curve of class C^3. If $S_1\varphi(x) \leq 2P(x)$ on $(-1,1)$, then φ is injective.*

4. Growth and distortion results for the metric

The first result of this section generalizes (2) to the subclass of holomorphic curves Ψ which satisfy (6) and $\partial_z\lambda(0) = 0$.

Lemma 1. *Let p be a Nehari's function and $\Psi : \mathbb{D} \to \mathbb{C}^n$ a holomorphic curve such that $\Psi'(z) \neq 0$ for all $z \in \mathbb{D}$. Suppose that Ψ satisfies (6) and $\partial_z\lambda(0) = 0$. Then*

$$\left|\frac{\partial \log \lambda}{\partial z}(z)\right| \leq w\left(|z|\right) \tag{10}$$

for all $z \in \mathbb{D}$, where w is the solution of the initial value problem

$$\begin{cases} w'(t) = w^2(t) + p(t), \\ w(0) = 0, \end{cases} \tag{11}$$

with $0 \leq t < 1$.

Proof. Since the quantities involved in (10) are invariant under rotations, it suffices to show (10) in the case $0 \leq z < 1$. On the other hand, if $y(t) = \partial_z \log\lambda(t)$, it follows that

$$\begin{aligned} y' &= \left\{\frac{1}{2}\frac{\langle \Psi''', \overline{\Psi'}\rangle}{\lambda^2} - \frac{1}{2}\frac{\langle \Psi'', \overline{\Psi'}\rangle^2}{\lambda^4}\right\} + \left\{\frac{1}{2}\frac{|\Psi''|^2}{\lambda^2} - \frac{1}{2}\frac{\langle \Psi'', \overline{\Psi'}\rangle\langle \Psi', \overline{\Psi''}\rangle}{\lambda^4}\right\} \\ &= (\log\lambda)_{zz} + (\log\lambda)_{z\bar{z}}. \end{aligned}$$

From here, and from the definition of $S\Psi$, we obtain that

$$y' = \frac{1}{2}S\Psi + y^2 + (\log\lambda)_{z\bar{z}},$$

or equivalently, by (3)

$$y' = y^2 + \left(\frac{1}{2}S\Psi - \frac{1}{4}\lambda^2 K\right). \tag{12}$$

Hence, by (6) and (12), we conclude that $\varphi(t) = |\,y(t)|$ satisfies

$$\begin{cases} \varphi'(t) \leq \varphi(t)^2 + p(t), \\ \varphi(0) = 0. \end{cases} \tag{13}$$

Comparing (13) with (11) we have

$$\begin{cases} (\varphi - w)'(t) \leq (\varphi - w)(\varphi + w)(t), \\ (\varphi - w)(0) = 0, \end{cases}$$

which implies that, for $0 \leq t < 1$,

$$[e^{-\int_0^t (\varphi + w) ds}(\varphi - w)]' = e^{-\int_0^t (\varphi + w) ds}[(\varphi - w)' - (\varphi - w)(\varphi + w)] \leq 0.$$

Hence, as $\varphi(0) = w(0)$, we can get $e^{-\int(\varphi+w)dt}(\varphi - w) \leq 0$ and so, $|y(t)| = \varphi(t) \leq w(t)$, for all $0 \leq t < 1$. □

Remark 1. In particular, for the Nehari's functions $p(t) = \left(1 - t^2\right)^{-2}$, $p(t) = 2\left(1 - t^2\right)^{-1}$, and $p(t) = \pi^2/4$ we have $w(t) = t/1 - t^2$, $w(t) = 2t/1 - t^2$, and $w(t) = \frac{\pi}{2}\tan\left(\frac{\pi t}{2}\right)$, respectively. Also, if $p(t) = \mu\left(1 - t^2\right)^{-2}$, $0 < \mu < 1$, then

$$w(t) = \frac{t}{1 - t^2} - \frac{2\alpha^2}{1 - t^2} A_\mu(t), \tag{14}$$

where $\alpha = \sqrt{1 - \mu}$ and A_μ is given by

$$A_\mu(z) = \frac{1}{\alpha} \frac{(1 + z)^\alpha - (1 - z)^\alpha}{(1 + z)^\alpha + (1 - z)^\alpha}.$$

Moreover, a straightforward calculation shows that A_μ is convex in $[0, 1]$, so $\psi(t) = \frac{A_\mu(t)}{t}$ is increasing here. Thus,

$$1 - \alpha^2 \frac{A_\mu(t)}{t} \leq 1 - \alpha^2 A'_\mu(0) = 1 - \alpha^2 = \mu.$$

From this, (14) and the inequality (10) it follows that if $\Psi \in Nh_0^\mu$, $0 < \mu \leq 1$,

$$\left|\frac{\partial \log \lambda}{\partial z}(z)\right| \leq \frac{\mu|z|}{1 - |z|^2} \tag{15}$$

for all $z \in \mathbb{D}$.

We illustrate Lemma 1 by using Example 2 in [4]:

Example. We consider the Nehari's functions $p_1(z) = \left(1 - z^2\right)^{-2}$ and $p_2(z) = 2\left(1 - z^2\right)^{-1}$. The analytic and univalent functions Φ_1 and Φ_2 in $\mathbb{D}$ given by

$$\Phi_1(z) = \frac{1}{2}\log\frac{1 + z}{1 - z} \quad \text{and} \quad \Phi_2(z) = \frac{1}{4}\log\frac{1 + z}{1 - z} + \frac{1}{2}\frac{z}{1 - z^2}$$

satisfy $S\Phi_j(z) = 2p_j(z)$, $j = 1, 2$. The image $\Phi_j(\mathbb{D})$ is a parallel strip like domain, symmetric with respect to the real and imaginary axes, and containing the entire real line. Let

$$\psi_j(z) = \frac{c\Phi_j(z) + i}{c\Phi_j(z) - i}, \qquad j = 1, 2$$

where $c > 0$ is to be chosen sufficiently small so that $i/c \notin \Phi_j(\mathbb{D})$. The functions ψ_j maps $\mathbb{D}$ onto a simply-connected domain containing the unit circle minus the point 1. Define $\Psi : \mathbb{D} \to \mathbb{C}^2$ by

$$\Psi_j(z) = \left(\psi_j, \frac{1}{\psi_j}\right), \qquad j = 1, 2.$$

A straightforward calculation shows that if $\lambda_j := \|\Psi_j'\|$ then $\partial_z \lambda_j(0) = 0$. Furthermore, according to Example 2 in [4] Ψ_j satisfies (6), namely Ψ_j verifies the hypotheses of Lemma 1.

We say that a function $u : [0,1) \to \mathbb{R}^+$ is *eventually increasing* if there is $x_0 \in [0,1)$ such that $u(t)$ is increasing in $x_0 \le t < 1$.

Lemma 2. *Let $\Psi \in Nh$ and suppose Ψ bounded. Then*

$$u_\Psi(z) := \frac{1}{\sqrt{\left(1 - |z|^2\right)\lambda(z)}} \tag{16}$$

has a critical point in $\mathbb{D}$.

Proof. Since Ψ is bounded we have that $\int_{-1}^{1} \lambda(t)\,dt < \infty$. Let $h(t) = \int_0^t \lambda(\tau)\,d\tau$ be the arc length function. It follows from Lemma 2 in [4] that

$$Sh(t) \le S_1\Psi(t) \le |S\Psi(t)| + \frac{3}{4}\lambda^2(t)\,|K(t)| \le \frac{2}{(1-t^2)^2}, \tag{17}$$

for $t \in (0,1)$. We define the function

$$x(s) = \frac{e^{2s} - 1}{e^{2s} + 1}, \quad -\infty < s < \infty,$$

which is bijective and increasing from $\mathbb{R}$ onto $(-1,1)$ with inverse $s(x) = \frac{1}{2}\log\frac{1+x}{1-x}$. Note that $v = (w \circ x)/\sqrt{x'}$ where $w = 1/\sqrt{h'}$, which is a solution of

$$w'' + \frac{1}{2}(Sh)\,w = 0. \tag{18}$$

From (18) and the equality $x' = 1 - x^2$, straightforward calculations produce

$$v''(s) = \left(1 - \left(1 - x^2(s)\right)^2 \frac{1}{2} Sh(x(s))\right) v(s).$$

It follows from (17) that the expression in parenthesis is non negative, so v is convex. We claim that u_Ψ has an absolute minimum. Indeed, since the integral

$$\int_0^1 h'(t)dt = \int_0^\infty h'(x(s))x'(s)ds = \int_0^\infty \frac{ds}{v^2(s)}$$

is finite then the convexity of v implies that $v(s) \to \infty$ when $s \to \infty$, and therefore v is eventually increasing which implies that $u_\Psi(t) \to \infty$ when $t \to 1$. Then, given $M > 0$, for all $\theta \in [0, 2\pi]$ there is $r_\theta \in [0,1)$ such that $u_\Psi(r) \ge M$, for all $r \ge r_\theta$. Let $r(M)$ be the maximum of r_θ that satisfies this condition, then $u_\Psi(r) \geq M$, for all $r \geq r(M)$ and for all $\theta \in [0, 2\pi]$. It follows that $u_\Psi(z) \to \infty$ when $|z| \to 1$. So u_Ψ has a minimum in $\mathbb{D}$ and therefore a critical point there. □

Corollary 1. *Let $\Psi \in Nh$. Then there is an absolute constant $M > 0$ such that*

$$\left|\frac{\partial \log \lambda}{\partial z}(z)\right| \leq \frac{M}{1-|z|^2},$$

for all $z \in \mathbb{D}$.

Proof. Suppose first that $\Psi(\mathbb{D})$ is bounded. By Lemma 2, there is a critical point $z_0 \in \mathbb{D}$ of the function u_Ψ defined in (16). With $\varphi(w) = \frac{z_0 - w}{1-\bar{z_0}w}$ we have that $g = \Psi \circ \varphi \in Nh$. Therefore, since $\varphi \in \mathrm{Aut}(\mathbb{D})$, we obtain from (4) that $u_g = u_\Psi \circ \varphi$, and consequently u_g has a critical point at zero. From here and the equality

$$\frac{\partial_w u_g}{u_g}(w) = \frac{1}{2}\left\{\frac{\bar{w}}{1-|w|^2} - \partial_w \log \lambda_g(w)\right\},$$

it follows that λ_g has a critical point at zero. We conclude that $g \in Nh_0$ and thus, by the inequality (15)

$$\left|\frac{\partial \log \lambda_g}{\partial w}(w)\right| \leq \frac{|w|}{1-|w|^2}, \qquad w \in \mathbb{D}. \tag{19}$$

On the other hand, as $\varphi = \varphi^{-1}$, $\Psi = g \circ \varphi$ and hence $\lambda_\Psi(z) = \lambda_g(\varphi(z))|\varphi'(z)|$. Thus

$$\partial_z \log \lambda_\Psi(z) = (\partial_w \log \lambda_g(\varphi(z)))\,\varphi'(z) + \frac{1}{2}\frac{\varphi''}{\varphi'}(z).$$

From definition of φ and (19), we obtain that

$$|\partial_z \log \lambda_\Psi(z)| \leq \frac{|\varphi(z)|}{1-|\varphi(z)|^2}|\varphi'(z)| + \left|\frac{\bar{z_0}}{1-\bar{z_0}z}\right| \leq \frac{3}{1-|z|^2}.$$

This proves the bounded case. The unbounded case follows by applying the above argument to $\Psi_r(z) = \frac{1}{r}\Psi(rz)$, $z \in \mathbb{D}$ and letting $r \to 1^-$. □

Corollary 2. *If $\Psi \in Nh_0^\mu$, $0 < \mu \leq 1$, then for all $\xi \in \mathbb{T}$ and $0 < r < 1$,*

$$\frac{1}{2^\mu}\lambda(r\xi) \leq \lambda(\rho\xi) \leq 2^\mu \lambda(r\xi),$$

$r \leq \rho \leq \frac{1+r}{2}$.

Proof. Given $\xi \in \mathbb{T}$ and $0 < r < \rho < 1$,

$$\log \frac{\lambda(\rho\xi)}{\lambda(r\xi)} = \int_r^\rho \frac{\partial}{\partial s}\log \lambda(s\xi)ds = \int_r^\rho 2\mathrm{Re}\left\{\partial_z \log \lambda(s\xi)\xi\right\}ds.$$

From Lemma 1 we have that

$$\left|\log \frac{\lambda(\rho\xi)}{\lambda(r\xi)}\right| \leq \int_r^\rho \frac{2\mu s}{1-s^2}ds \leq \mu \log 2$$

if $r \leq \rho \leq \frac{1+r}{2}$, and the corollary follows. □

Remark 2. Under the same hypothesis of Corollary 2, we have that

$$\frac{1}{M}\lambda(r\xi) \leq \lambda(\rho\xi) \leq M\lambda(r\xi)$$

if $0 < r \leq \rho < 1$ satisfy $1 - r^2 \leq M(1 - \rho^2)$, with M an absolute constant.

The following corollary is established in analogous form.

Corollary 3. *Let $\Psi \in Nh_0$ and $k > 0$. If $z = re^{i\theta}$ and $\zeta = re^{i\nu}$ with $0 < r < 1$ and $|\theta - \nu| \leq k(1 - r)$, then*

$$e^{-2k}\,\lambda(\zeta) \leq \lambda(z) \leq e^{2k}\,\lambda(\zeta).$$

Remark 3. If Ψ is a holomorphic curve satisfying (6) with $p(t) = (1-t^2)^{-2}$, then (see [4], Theorem 4)

$$d_\Psi(z_0) \geq \frac{(1 - |z_0|^2)\lambda(z_0)}{\sqrt{2}\,(1 - |z_0|) + \left|(1 - |z_0|^2)(\partial_z \log \lambda)(z_0) - \overline{z_0}\right|}, \quad z_0 \in \mathbb{D}, \tag{20}$$

where $d_\Psi(z_0) := d\,(\Psi(z_0), \partial\Psi(\mathbb{D}))$. It follows from this and Corollary 1 that $C = 4 + \sqrt{2}$ satisfies

$$(1 - |z_0|^2)\lambda(z_0) \leq C\, d_\Psi(z_0), \quad z_0 \in \mathbb{D}, \tag{21}$$

for all $\Psi \in Nh$.

5. Proof of the main theorems

We start with some preliminary lemmas found in [14].

Lemma 3. *Let g be a real smooth function on $(-1, 1)$ such that $g'(t) > 0$ for all $t \in (-1, 1)$, and $g''(0) = 0$. Suppose that the Schwarzian derivative of g satisfies*

$$Sg(t) \leq \frac{2\mu}{(1 - t^2)^2}, \qquad 0 \leq \mu < 1$$

on $(-1, 1)$. Then there is $M > 0$ (depending only on μ) such that

$$\int_r^1 g'(t)\,dt \leq M\left(1 - r^2\right) g'(r), \qquad 0 \leq r < 1 \tag{22}$$

and

$$\int_{-1}^r g'(t)\,dt \leq M\left(1 - r^2\right) g'(r), \qquad -1 < r \leq 0. \tag{23}$$

Lemma 4. *Let $a = re^{i\theta} \in \mathbb{D}$ and S be the hyperbolic segment orthogonal to diameter $[-e^{i\theta}, e^{i\theta}]$ at a. Let $e^{i\theta_1}$ and $e^{i\theta_2}$, $\theta_1 < \theta_2$ be the endpoints of S. There is a constant K independent of r, such that for all $w = r_1e^{i\alpha} \in S$ it holds*

(i) $|\theta_2 - \alpha| \leq K(1 - r_1)$, *if* $\theta \leq \alpha$;
(ii) $|\theta_1 - \alpha| \leq K(1 - r_1)$, *if* $\alpha \leq \theta$.

Remark 4. With the notation before, we conclude from Lemma 4 and Corollary 3 that there exists $M > 0$ such that if $\Psi \in Nh_0$, then

(i) $\frac{1}{M}\lambda(r_1 e^{i\theta_2}) \leq \lambda(w) \leq M\lambda(r_1 e^{i\theta_2})$, if $\theta \leq \alpha$,

(ii) $\frac{1}{M}\lambda(r_1 e^{i\theta_1}) \leq \lambda(w) \leq M\lambda(r_1 e^{i\theta_1})$, if $\theta \geq \alpha$.

The proof of the following lemma is similar to the proof of Lemma 6 in [14], and is included here for the convenience of the reader.

Lemma 5. *Let a and S be as in Lemma 4 and σ the automorphism of the unit disk that maps $(-1, 1)$ onto S in such a way that $\sigma(-1) = e^{i\theta_2}$, $\sigma(0) = a$ and $\sigma(1) = e^{i\theta_1}$. Suppose that $\Psi \in Nh_0^\mu$, $0 < \mu < 1$ and $\lambda_1 = (\lambda \circ \sigma) |\sigma'|$. If x is a critical point of the function*

$$v(t) = \frac{1}{\sqrt{(1-t^2)\lambda_1(t)}}, \qquad -1 < t < 1 \tag{24}$$

and $|x| > \mu + \eta$, for some $\eta > 0$, then there is $M > 0$ such that

$$\frac{1}{M}\lambda(y) \leq \lambda(a) \leq M\lambda(y),$$

where $y = \sigma(x)$.

Proof. By a straightforward calculation one can see that $\sigma(z) = -ie^{i\theta}\frac{ri+z}{1-riz}$,

$$\frac{v'(t)}{v(t)} = \frac{1}{2}\left\{\frac{2t}{1-t^2} - \frac{\lambda_1'(t)}{\lambda_1(t)}\right\},$$

and

$$\frac{\lambda_1'(t)}{\lambda_1(t)} = \frac{\langle \nabla\lambda(\sigma(t)), \sigma'(t)\rangle}{\lambda(\sigma(t))} + \mathrm{Re}\frac{\sigma''(t)}{\sigma'(t)}.$$

From this and the assumption $v'(x) = 0$ we obtain

$$\frac{2x}{1-x^2} = \frac{\langle \nabla\lambda(\sigma(x)), \sigma'(x)\rangle}{\lambda(\sigma(x))} + \mathrm{Re}\frac{\sigma''(x)}{\sigma'(x)}.$$

Hence, by (15) and $\mathrm{Re}\frac{\sigma''(x)}{\sigma'(x)} = -\frac{2r^2x}{1+r^2x^2}$, we deduce that

$$\begin{aligned}\frac{|x|}{1-|x|^2} &\leq \frac{\mu|\sigma(x)|}{1-|\sigma(x)|^2}|\sigma'(x)| + \frac{r^2|x|}{1+r^2x^2} \leq \frac{\mu|\sigma(x)|}{1-|x|^2} + \frac{1}{1+r^2x^2}\\ &\leq \frac{\mu}{1-|x|^2} + \frac{1}{1+r^2x^2},\end{aligned}$$

and so

$$\frac{|x|-\mu}{1-|x|^2} \leq \frac{1}{1+r^2x^2}.$$

We conclude from the condition $|x| > \mu + \eta$ that

$$\frac{\eta}{1-x^2} < \frac{1}{1+r^2x^2}$$

and therefore $\frac{1+r^2x^2}{1-x^2} < \frac{1}{\eta}$. Finally, the equalities

$$\frac{|\sigma'(x)|}{1-|\sigma(x)|^2} = \frac{1}{1-|x|^2} \quad \text{and} \quad |\sigma'(x)| = \frac{1-r^2}{1+r^2x^2},$$

imply

$$\frac{1-|a|^2}{1-|\sigma(x)|^2} = \frac{1+r^2x^2}{1-x^2} \le \frac{1}{\eta}. \tag{25}$$

The lemma now follows from Remarks 2 and 4. □

From Lemma 5 and the inequality (25) we obtain the following result.

Corollary 4. *Under the hypothesis of Lemma 5, there exist positive constants $C = C(\mu,\eta)$ and $\delta = \delta(\mu,\eta)$ such that*

$$\delta \le \frac{(1-|y|^2)\lambda(y)}{(1-|a|^2)\lambda(a)} \le C,$$

where $y = \sigma(x)$.

Proof of Theorem 2. Step 1. Suppose Ψ is bounded. By Corollary 1 we can assume that $\nabla\lambda(0) = 0$. Let $A, B \in \Psi(\mathbb{D}) = \Sigma$ and $a, b \in \overline{\mathbb{D}}$ such that $\Psi(a) = A$ and $\Psi(b) = B$. We consider the curve $\Gamma = \Psi(S)$, where $S := S(a,b)$ is the hyperbolic segment with endpoints a and b. We will prove that given $p \in \Gamma$

$$\min\{l(\Gamma(p,A)), l(\Gamma(p,B))\} \le M_\mu d(p,\partial\Sigma).$$

Let $q \in \mathbb{D}$ such that $\Psi(q) = p$. Without loss of generality, we can assume that if $z_0 = r_0e^{i\theta_0}$ is the midpoint of the hyperbolic geodesic γ passing through a and b, then $\mathrm{Arg}a > \mathrm{Arg}z_0$ and $q \in S(z_0,a)$. A parametrization of $S(q,a)$ is given by

$$\varphi(t) = -ie^{i\theta_0}\frac{r_0i - t}{1+r_0it}; \qquad t_1 \le t \le t_2, \quad t_1, t_2 \in [0,1).$$

Hence, by Remark 3, there is a constant k such that

$$l(\Psi(S(q,a))) \le \int_{t_1}^1 \lambda(\varphi(t))|\varphi'(t)|dt \le k\int_{t_1}^1 \lambda(|\varphi(t)|e^{i\theta})|\varphi'(t)|dt, \tag{26}$$

where $\theta > \mathrm{Arg}\{a\}$ and $e^{i\theta}$ is an endpoint of γ. Consider two cases: if $t_1 \ge 1/2$, defining $u = |\varphi(t)|$ we have that

$$uu' = t\frac{1+r^2}{1+r^2t^2}|\varphi'(t)| \quad \text{and} \quad t^2 = \frac{u^2-r^2}{1-r^2u^2},$$

whence

$$\left|\varphi'(t)\right| = \frac{1}{t(1+r^2)}\sqrt{1+r^2t^2}\,\sqrt{t^2+r^2}\,u' \le 4u',$$

$1/2 \le t_1 \le t \le 1$. Thus, by (26), Remark 2 and (21) we obtain

$$l(\Psi(S(q,a))) \le 4k\int_{|\varphi(t_1)|}^1 \lambda(ue^{i\theta})du \le C\left(1-|\varphi(t_1)|^2\right)\lambda(\varphi(t_1)) \le \widetilde{M}d_\Psi(q),$$

where the constants depend only on μ. Now, if we suppose that $0 \le t_1 < 1/2$, it follows from the above that

$$\begin{aligned} l\left(\Psi\left(S\left(q,a\right)\right)\right) &\le \int_{t_1}^{1/2} \lambda(\varphi(t))\left|\varphi'(t)\right| dt + \int_{1/2}^{1} \lambda(\varphi(t))\left|\varphi'(t)\right| dt \\ &\le \int_{t_1}^{1/2} \lambda(\varphi(t))\left|\varphi'(t)\right| dt + 4k \int_{|\varphi(1/2)|}^{1} \lambda(ue^{i\theta}) du, \end{aligned}$$

since $|\varphi(t_1)| \le |\varphi(1/2)|$. On the other hand, for $0 \le t \le \frac{1}{2}$

$$1 - |\varphi(t)|^2 \ge 1 - |\varphi(1/2)|^2 = 1 - \frac{r_0^2 + \left(\frac{1}{2}\right)^2}{1 + \left(\frac{r_0}{2}\right)^2},$$

whence $1 - |\varphi(t_1)|^2 \le 1 - r_0^2 \le 2(1 - |\varphi(t)|^2)$, $0 \le t \le \frac{1}{2}$. We conclude from here and Remark 2 that there is M' such that

$$\begin{aligned} \int_{t_1}^{1/2} \lambda(\varphi(t))\left|\varphi'(t)\right| dt &\le M' \int_{t_1}^{1/2} \lambda(|\varphi(t)|e^{i\theta})\left|\varphi'(t)\right| dt \\ &\le M' \int_{t_1}^{1/2} \lambda(\left|\varphi(t_1)\right| e^{i\theta})\left|\varphi'(t)\right| dt. \end{aligned}$$

Thus, from $|\varphi'(t)| = \frac{1-r_0^2}{1+r_0^2t^2} \le 1 - r_0^2 \le 2(1 - |\varphi(t_1)|^2)$, Remark 3 and (21) it follows

$$\begin{aligned} \int_{t_1}^{1/2} \lambda(\varphi(t))\left|\varphi'(t)\right| dt &\le 2M'\left(1 - |\varphi(t_1)|^2\right) \lambda(|\varphi(t_1)|e^{i\theta}) \\ &\le M''\left(1 - |\varphi(t_1)|^2\right) \lambda(\varphi(t_1)) \le k'' d_\Psi(q). \end{aligned}$$

This proves the condition (ii) of Definition 1 in the bounded case. Note that k'' only depends on μ.

Step 2. We will show that there is $M > 0$ such that for all $\Psi \in Nh^\mu$ bounded and for all $a, b \in \partial\mathbb{D}$

$$l\left(\Psi\left(S\right)\right) \le M \left\|\Psi\left(a\right) - \Psi\left(b\right)\right\|, \tag{27}$$

where S is the hyperbolic segment with endpoints a and b. If there is no such constant M, then for all $n > 0$, there is $f_n \in Nh^\mu$ bounded and $\zeta_n^\pm \in \mathbb{T}$ such that

$$l\left(f_n\left(S_n\right)\right) \ge n \left\|f_n\left(\zeta_n^+\right) - f_n\left(\zeta_n^-\right)\right\|, \tag{28}$$

here S_n is the hyperbolic segment with endpoints ζ_n^+ and ζ_n^-. Without loss of generality, we can assume that $\nabla\lambda_{f_n}(0) = 0$. Proceeding as in Step 1 one can show that

$$l\left(f_n\left(S_n\right)\right) \le C\left(1 - |z_n|^2\right) \lambda_{f_n}\left(z_n\right),$$

where z_n is the midpoint of S_n and C only depends on μ. It follows from here and the inequality (28) that

$$\lim_{n\to\infty} \frac{\|f_n(\zeta_n^+) - f_n(\zeta_n^-)\|}{(1-|z_n|^2)\lambda_{f_n}(z_n)} = 0. \tag{29}$$

Given n let $\sigma_n \in \mathrm{Aut}(\mathbb{D})$ be such that $\sigma_n(\pm 1) = \zeta_n^\pm$ and $\sigma_n(0) = z_n$. Note that $\sigma_n(z) = -ie^{i\theta_n}\frac{ir_n+z}{1-ir_nz}$ if $z_n = r_ne^{i\theta_n}$, $f_n \circ \sigma_n \in Nh^\mu$, and $\lambda_{f_n\circ\sigma_n} = (\lambda_{f_n} \circ \sigma_n)|\sigma_n'|$. From Lemma 2, we can conclude that the functions

$$u_{f_n\circ\sigma_n}(t) = \frac{1}{\sqrt{(1-t^2)\lambda_{f_n\circ\sigma_n}(t)}}, \qquad 0 < t < 1$$

have an absolute minimum at some $x_n \in (-1,1)$.

Now we consider the sequence of functions $F_n = f_n \circ R_n$, where $R_n = \sigma_n \circ Q_n$ and $Q_n(z) = \frac{x_n - z}{1-x_nz}$. It is easy to see that $F_n \in Nh^\mu$,

$$F_n(\pm 1) = f_n(\zeta^\pm) \quad \text{and} \quad \lambda_{F_n} = (\lambda \circ R_n)|R_n'|.$$

From the discussion above it follows that $u_{F_n}(t) = (u_{f_n\circ\sigma_n} \circ Q_n)(t)$ has a critical point at $t = 0$ and hence $\frac{\partial\lambda_{F_n}}{\partial x}(0) = 0$. We conclude that

$$\Psi_n(z) = \frac{F_n(z) - F_n(0)}{(1-|y_n|^2)\lambda_{f_n(y_n)}}, \qquad y_n = T_n(x_n)$$

satisfies $\Psi_n \in Nh^\mu$, $\lambda_{\Psi_n}(0) = 1$, $\partial_x\lambda_{\Psi_n}(0) = 0$ and by Corollary 1

$$\left|\frac{\partial \log \lambda_{\Psi_n}}{\partial z}(z)\right| \le \frac{3}{1-|z|^2}.$$

Integrating we obtain

$$\frac{(1-|z|)^3}{8} \le \lambda_{\Psi_n}(z) \le \frac{8}{(1-|z|)^3}; \qquad z \in \mathbb{D}. \tag{30}$$

Since the components of Ψ_n are analytic in $\mathbb{D}$, it follows from (30) that Ψ_n has a subsequence Ψ_{n_k}, that we denote Ψ_n, which converges locally uniformly in $\mathbb{D}$ to a holomorphic curve $\Psi \in Nh^\mu$. Note that if $\lambda(z) = \|\Psi'(z)\|$, $z \in \mathbb{D}$, then $\lambda_{\Psi_n} \to \lambda$ locally uniformly in $\mathbb{D}$. Also, for all $j = 1, 2, ...$

$$\frac{\partial^j\lambda_{\Psi_n}}{\partial x^j}(t) \to \frac{\partial^j\lambda}{\partial x^j}(t),$$

for all $t \in (-1,1)$ and accordingly $S_1\Psi_n(t) \to S_1\Psi(t)$. From (9) and the inequality (see [4], Lemma 2)

$$S_1\Psi_n(t) = Ss_n(t) + \lambda_{\Psi_n}^2(t)k_n^2(t) \le \frac{2\mu}{(1-t^2)^2}, \quad -1 < t < 1,$$

we obtain

$$S_1\Psi(t) \le \frac{2\mu}{(1-t^2)^2} \quad \text{and} \quad Ss_n(t) \le \frac{2\mu}{(1-t^2)^2}, \quad -1 < t < 1,$$

where $s_n(t) = \int_0^t \lambda_{\Psi_n}(\nu)\, d\nu$ and $k_n(t)$ is the curvature of $\Psi_n(t)$. Since $s_n''(0) = \partial_x \lambda_{\Psi_n}(0) = 0$, we can argue as in the proof of Lemma 1 (see also [14], Lemma 2), and conclude that

$$\frac{s_n''(t)}{s_n'(t)} \leq \mu \frac{2t}{1-t^2}, \qquad 0 \leq t < 1$$

and therefore

$$\lambda_{\Psi_n}(t) = s_n'(t) \leq \frac{1}{(1-t^2)^{\mu}}, \qquad 0 \leq t < 1. \tag{31}$$

By applying the results derived above to $\tilde{s}_n(t) = -s_n(-t)$, $-1 < t < 1$, we conclude that the inequality (31) also holds for $-1 < t < 0$, since $\tilde{s}_n$ and s_n have the same Schwarzian derivative and $\tilde{s}_n''(0) = s''(0) = 0$. Then the sequence Ψ_n is equicontinuous and bounded in $[-1,1]$, which implies that Ψ_n converges to Ψ uniformly in $[-1,1]$. In consequence

$$\|\Psi_n(-1) - \Psi_n(1)\| \to \|\Psi(-1) - \Psi(1)\|.$$

Now, by definition of Ψ_n,

$$\begin{aligned} \|\Psi_n(-1) - \Psi_n(1)\| &= \frac{\|f_n(\zeta_n^+) - f_n(\zeta_n^-)\|}{(1-|y_n|^2)\,\lambda_{f_n}(y_n)} \\ &= \frac{\|f_n(\zeta_n^+) - f_n(\zeta_n^-)\|}{(1-|z_n|^2)\,\lambda_{f_n}(z_n)} \frac{\left(1-|z_n|^2\right)\lambda_{f_n}(z_n)}{(1-|y_n|^2)\,\lambda_{f_n}(y_n)}. \end{aligned} \tag{32}$$

From Corollary 4 it follows that there is a constant $\beta > 0$ such that

$$\frac{\left(1-|z_n|^2\right)\lambda_{f_n}(z_n)}{(1-|y_n|^2)\,\lambda_{f_n}(y_n)} \leq \beta.$$

Then, from (29) and (32) we have $\|\Psi_n(-1) - \Psi_n(1)\| \to 0$ and so $\Psi(-1) = \Psi(1)$ which is a contradiction since, from Theorem 1, Ψ is injective in $\overline{\mathbb{D}}$. This proves (27).

Step 3. We prove that $\Psi(\mathbb{D})$ is an uniform surface for $\Psi \in Nh^{\mu}$. Given $A, B \in \Psi(\mathbb{D})$, $A \neq B$, there are $a, b \in \mathbb{D}$ such that $\Psi(a) = A$ and $\Psi(b) = B$. We will show that $\Gamma := \Psi(S(a,b))$ satisfies (i) and (ii) of Definition 1. If we compose with an automorphism of $\mathbb{D}$ we can assume that $|a| = |b| = \rho$. The function $\Psi_\rho(z) = \Psi(\rho z)$ satisfies $\Psi_\rho \in Nh^{\mu}$, $\Psi_\rho(\mathbb{D})$ is bounded, $A = \Psi_\rho(a/\rho)$, $B = \Psi_\rho(b/\rho)$, $A, B \in \partial\Psi_\rho(\mathbb{D})$ and $\Gamma = \Psi_\rho(S(a/\rho, b/\rho))$. It follows from Step 2 that

$$l(\Gamma) = l(\Psi_\rho(S(a/\rho, b/\rho))) \leq M_\mu \|\Psi_\rho(a/\rho) - \Psi_\rho(b/\rho)\| = M_\mu \|A - B\|;$$

which prove (i). To show (ii) we take $p \in \Gamma$, there is $q \in S(a,b)$ such that $\Psi_\rho(q/\rho) = \Psi(q) = p$. We conclude from Step 1 that

$$\begin{aligned} \min\{l(\Gamma(p,A)), l(\Gamma(p,B))\} &\leq k_\mu d_{\Psi_\rho}(q/\rho) = k_\mu d(p, \partial\Psi_\rho(\mathbb{D})) \\ &\leq k_\mu d(p, \partial\Psi(\mathbb{D})) = k_\mu d_\Psi(q). \end{aligned}$$

□

Proof of Theorem 3. By Lemma 2 in [4], the curve $\phi := \Psi|_{(-1,1)}$ satisfies

$$S_1\phi(x) = \operatorname{Re}\{S\Psi(x)\} + \frac{3}{4}\lambda^2(x)|K(\phi(x))|, \qquad -1 < x < 1.$$

The condition $\Psi \in Nh$ implies

$$S_1\phi(x) \leq \frac{2}{(1-x^2)^2}, \qquad -1 < x < 1,$$

and hence by Theorem 2 in [5],

$$\frac{\|\phi(x_1) - \phi(x_2)\|}{\{\|\phi'(x_1)\|\|\phi'(x_2)\|\}^{1/2}} \geq \sqrt{(1-x_1^2)(1-x_2^2)}\, d_h(x_1,x_2), \qquad x_1, x_2 \in (-1,1)$$

or equivalently,

$$\frac{\|\Psi(x_1) - \Psi(x_2)\|}{\{\lambda(x_1)\lambda(x_2)\}^{1/2}} \geq \sqrt{(1-x_1^2)(1-x_2^2)}\, d_h(x_1,x_2), \qquad x_1, x_2 \in (-1,1).$$

In the general case, given $z_1, z_2 \in \mathbb{D}$, there are $T \in \operatorname{Aut}(\mathbb{D})$ and points $x_1, x_2 \in (-1,1)$ such that $T(x_1) = z_1$ and $T(x_2) = z_2$. Since $\Psi \circ T \in Nh$, then

$$\frac{\|\Psi(T(x_1)) - \Psi(T(x_2))\|}{\{\lambda_{\Psi\circ T}(x_1)\lambda_{\Psi\circ T}(x_2)\}^{1/2}} \geq \sqrt{(1-x_1^2)(1-x_2^2)}\, d_h(x_1,x_2).$$

It follows from here and the equality $\lambda_{\Psi\circ T} = (\lambda \circ T)\,|T'|$ that

$$\frac{\|\Psi(z_1) - \Psi(z_2)\|}{\{\lambda(z_1)\lambda(z_2)\}^{1/2}} \geq \sqrt{(1-x_1^2)|T'(x_1)|(1-x_2^2)|T'(x_2)|}\, d_h(x_1,x_2)$$

and consequently

$$\frac{\|\Psi(z_1) - \Psi(z_2)\|}{\{\lambda(z_1)\lambda(z_2)\}^{1/2}} \geq \sqrt{(1-|z_1|^2)(1-|z_2|^2)}\, d_h(z_1,z_2),$$

which shows (7).

On the other hand, since $\Psi \in Nh_0^\mu$, we obtain from Lemma 1 that

$$|\nabla \log \lambda(z)| = |2\partial_z \log \lambda(z)| \leq 2w(|z|), \tag{33}$$

where w is solution of the initial value problem

$$w'(t) = w^2(t) + \frac{\mu}{(1-t^2)^2}, \qquad w(0) = 0, \qquad 0 \leq t < 1.$$

Now, given $r \in (0,1)$ and $\theta \in [0, 2\pi)$,

$$\begin{aligned} \log \frac{\lambda(re^{i\theta})}{\lambda(0)} &= \int_0^r \frac{d}{dt} \log \lambda(te^{i\theta})dt \\ &= \int_0^r \left\langle \nabla \log \lambda(te^{i\theta}), e^{i\theta} \right\rangle dt \end{aligned}$$

and therefore, by (33),

$$\left|\log \frac{\lambda(re^{i\theta})}{\lambda(0)}\right| \leq \int_0^r 2w(t)dt.$$

Since $\mu \in (0,1)$ we have that

$$w(t) = \frac{t}{1-t^2} - \frac{\alpha^2}{1-t^2} A_\mu(t),$$

where

$$A_\mu(z) = \frac{1}{\alpha}\frac{(1+z)^\alpha - (1-z)^\alpha}{(1+z)^\alpha + (1-z)^\alpha}; \quad \alpha = \sqrt{1-\mu}\,.$$

It follows that

$$\begin{aligned}\left|\log \frac{\lambda(re^{i\theta})}{\lambda(0)}\right| &\le \int_0^r \frac{2t}{1-t^2}dt - \int_0^r \frac{2\alpha}{1-t^2}\frac{(1+t)^\alpha - (1-t)^\alpha}{(1+t)^\alpha + (1-t)^\alpha}dt \\ &= \log\frac{1}{1-r^2} - \int_0^r \frac{2\alpha}{1-t^2}\frac{1-\left(\frac{1-t}{1+t}\right)^\alpha}{1+\left(\frac{1-t}{1+t}\right)^\alpha}dt.\end{aligned}$$

The substitution $u = \left(\frac{1-t}{1+t}\right)^\alpha$ leads to

$$\left|\log \frac{\lambda(re^{i\theta})}{\lambda(0)}\right| \le \log\frac{1}{1-r^2} + \int_1^{u(r)} \frac{1}{u}\frac{1-u}{1+u}du = \log\frac{1}{1-r^2} + \int_1^{u(r)} \left(\frac{1}{u} - \frac{2}{1+u}\right)du,$$

therefore

$$\begin{aligned}\left|\log \frac{\lambda(re^{i\theta})}{\lambda(0)}\right| &\le \log\left[\frac{4}{1-r^2}\frac{u(r)}{(1+u(r))^2}\right] \\ &= \log\left[4\frac{(1-r)^{\alpha-1}(1+r)^{\alpha-1}}{((1+r)^\alpha + (1-r)^\alpha)^2}\right]\end{aligned}$$

and thus we conclude that

$$\lambda(re^{i\theta}) \le \lambda(0)\frac{4^{1-\alpha}}{(1-r)^{1-\alpha}}.$$

We obtain (8) by following the argument found in [12] (see also [8]). $\square$

We may also obtain an inequality type (8) assuming $\Psi \in Nh$ and Ψ bounded.

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